Wave scattering from self-affine surfaces

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Electromagnetic wave scattering from a perfectly reflecting self-affine surface is considered. Within the framework of the Kirchhoff approximation, we show that the scattering cross section can be exactly written as a function of the scattering angle via a centered symmetric Lévy distribution for the general roughness amplitude, Hurst exponent and wavelength of the incident wave. Our prediction is supported by direct numerical simulations.

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Wave scattering from rough surfaces has been studied for a long time [1,2] (see also [3–5] for recent reviews of the subject) with potential applications in remote sensing, acoustical/optical/radar detection or design of surfaces with specified scattering properties, etc. The theoretical prediction of the angular distribution of the scattered intensity requires a proper statistical description of the surface roughness. At this stage it is classical to assume that the statistics of the height and its correlation function are Gaussian. This assumption allows us to build analytical expressions and to compare them with experimental or numerical results.

Since the pioneering work of Mandelbrot [6], however, scale invariance has emerged as a relevant tool to describe the geometry of real "disordered" objects. In the case of surfaces, the scale invariance takes the form of self-affinity. Such surfaces remain invariant under dilation of different ratios over the horizontal and the vertical directions. This long-range correlated height distribution is characterized by a roughness exponent, an amplitude parameter, and by the lower and upper limits of the scaling invariance region (instead of a height standard deviation and a two-point correlation function of finite width). It turns out that many real surfaces can be described through this formalism. Surfaces obtained from fracture [7], growth, or deposition processes [8] are classical examples. Similar results have been more recently found for surfaces obtained from cold metal rolling [9].

Since an early paper examining the effect of scale invariance in the context of scattering from rough surfaces [10], a large amount of studies have been published in various journals (see, for example, Refs. [11–19]). Two main points motivated most of these studies: (i) the effect of scale invariance on the field scattered from the surface in comparison with the more classical case of a finite width correlation length, (ii) the hope of directly measuring the "fractal" parameters of the surface via an acoustical or optical experiment. In spite of a large amount of work, very few analytical predictions can be found and conversely a large number of numerical simulations still lack a clear physical interpretation. One reason for this could be an excessive focus on the "fractal dimension" (or the roughness exponent) with a disregard for the other parameters (the limits of the scale invariance regime and the amplitude parameter). A noteworthy exception is, however, due to Jakeman and his collaborators [20,21], who predicted a Lévy distribution for the intensity of a wave scattered from a self-affine random phase screen. In the following, we briefly recall the main properties of self-affine surfaces, give a complete analytic expression of the scattered intensity in the framework of the Kirchhoff approximation, and present direct numerical simulation that supports our predictions.

A surface is self-affine between the scales ξ_- and ξ_+ if it remains (either exactly or statistically) invariant in this region under transformations of the form

$$(x,y,z)\rightarrow(\lambda x,\lambda y,\lambda^H z). \tag{1}$$

The exponent H is usually called the roughness or Hurst exponent. Restricting ourselves to profiles, a statistical translation of the previous statement is that the probability $p(\Delta z;\Delta x)$ of having a height difference Δz over the distance Δx or its cumulative $\mathcal{P}(\Delta z;\Delta x)=\int_{-\infty}^{\Delta z}p(\delta z;\Delta x)d\,\delta z$ is such that

$$\mathcal{P}(\Delta z;\Delta x)=\mathcal{P}(\lambda^H\Delta z;\lambda\Delta x)=\Phi\left(\frac{\Delta z}{\Delta x^H}\right). \tag{2}$$

Simple algebra based on this scaling gives

$$\sigma(\Delta x)=l^{1-H}\Delta x^H, \tag{3}$$

where $\sigma(\Delta x)$ is the standard deviation of the height differences over a length Δx. Here l denotes a length scale, also known as the *topothesy*. This quantity is defined by $\sigma(l)=l$, which allows the geometrical interpretation of the topothesy as the length scale over which the profile has a mean slope of 45°. The smaller l, the flatter the profile appears on a macroscopic scale. In the case of a Gaussian height distribution, the probability $p(\Delta z;\Delta x)$ reads

$$p(\Delta z;\Delta x)=\frac{1}{\sqrt{2\pi}l^{1-H}\Delta x^H}\exp\left[-\frac{1}{2}\left(\frac{\Delta z}{l^{1-H}\Delta x^H}\right)^2\right]. \tag{4}$$

The self-affine profile is thus fully characterized by its exponent H, its topothesy parameter l, and the bounds of the self-affine regime ξ_- and ξ_+. Note that l can be outside the range $[\xi_-;\xi_+]$.

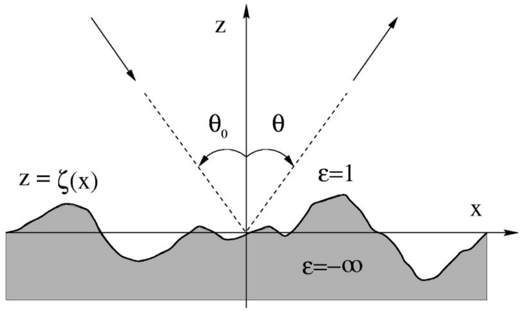

FIG. 1. Sketch of the scattering geometry.

In the following, we consider the scattering of *s*-polarized electromagnetic waves from a one-dimensional, random, self-affine surface. This surface, of Hurst exponent H, is denoted by $z=\zeta(x)$, and is assumed to be Gaussian self-affine. We will further assume that the lower limit of the self-affine regime ξ_- is smaller than the wavelength, λ, of the incident wave. The scattering geometry considered is depicted in Fig. 1. The incident plane is assumed to be the xz-plane, and the rough surface, which is perfectly conducting, is illuminated from the vacuum side by a plane wave of pulsation $\omega=2\pi c/\lambda$, where c is the wave velocity. We denote the incident and scattering angle by θ_0 and θ, respectively, and they are defined positive according to the convention indicated in Fig. 1. In the above scattering geometry, there is no depolarization, and the electromagnetic field is represented by the single nonvanishing component of the electric field $\Phi(x,z|\omega)=E_y(x,z|\omega)$, which should satisfy the (scalar) Helmholtz equation with vanishing boundary condition on $z=\zeta(x)$ and outgoing wave condition at infinity. In the far-field region, above the surface, the field can be represented as the sum of an incident wave and scattered waves:

$$\Phi(x,z|\omega)=\Phi_0(x,z|\omega)+\int_{-\infty}^{\infty}\frac{dq}{2\pi}R(q|k)e^{iqx+i\alpha_0(q,\omega)z}, \tag{5}$$

where the plane incident wave is given by

$$\Phi_0(x,z|\omega)=\exp\{ikx-i\alpha_0(k,\omega)z\} \tag{6}$$

and $R(q|k)$ is the *scattering amplitude*. In the above expression, we have defined $\alpha_0(q,\omega)=\sqrt{(\omega/c)^2-q^2}[\mathrm{Re}\,\alpha_0(q,\omega)>0,\mathrm{Im}\,\alpha_0(q,\omega)>0]$. Furthermore, the momentum variables q and k are in the radiative region related, respectively, to the scattering and incident angle by $q=(\omega/c)\sin\theta$ and $k=(\omega/c)\sin\theta_0$, so that $\alpha_0(q,\omega)=(\omega/c)\cos\theta$ and $\alpha_0(k,\omega)=(\omega/c)\cos\theta_0$.

The mean differential reflection coefficient (DRC) also known as the mean scattering cross section, which is an experimentally accessible quantity, and defined as the fraction of the total, time-averaged, incident energy flux scattered into the angular interval $(\theta,\theta+d\theta)$, is related to this quantity by the following expression [22]:

$$\left\langle\frac{\partial R}{\partial\theta}\right\rangle=\frac{1}{L}\,\frac{\omega}{2\pi c}\,\frac{\cos^2\theta}{\cos\theta_0}\langle|R(q|k)|^2\rangle. \tag{7}$$

Here L denotes the length of the self-affine profile function as measured along the x direction, and the other quantities have been defined earlier. The angular brackets denote an ensemble average over the surface profiles $\zeta(x)$, and the momentum variables are understood to be related to the angles θ_0 and θ according to the expressions given above.

We now use the Kirchhoff approximation, which consists of locally replacing the surface by its tangential plane, and thereafter using the (local) Fresnel reflection coefficient for the local incident angle to obtain the scattered field. Notice here that dealing with a surface whose scaling invariance range is bounded by a lower cutoff ξ_- does ensure that the tangential plane is well defined in every point. The scattering amplitude is then [22]

$$R(q|k)=\frac{-i}{2\alpha_0(q,\omega)}\int_{-L/2}^{L/2}dx\,e^{-iqx-i\alpha_0(q,\omega)\zeta(x)}\mathcal{N}(x|\omega), \tag{8}$$

where $\mathcal{N}(x|\omega)$ is a source function defined by $\mathcal{N}(x|\omega)=2\partial_n\Phi_0(x,z|\omega)|_{z=\zeta(x)}$, with the (unnormalized) normal derivative given by $\partial_n=-\zeta'(x)\partial_x+\partial_z$. The ensemble average then gives

$$\left\langle\frac{\partial R}{\partial\theta}\right\rangle=\frac{\omega}{2\pi c}\,\frac{1}{\cos\theta_0}\left(\frac{\cos[(\theta+\theta_0)/2]}{\cos[(\theta-\theta_0)/2]}\right)^2 \times\int_{-L/2}^{L/2}dv\,\exp\left\{i\frac{\omega}{c}(\sin\theta-\sin\theta_0)v\right\}\Omega(v), \tag{9}$$

where

$$\Omega(v)=\left\langle\exp\left\{-i\frac{\omega}{c}[\cos\theta+\cos\theta_0]\Delta\zeta(v)\right\}\right\rangle, \tag{10}$$

with $\Delta\zeta(v)=\zeta(x)-\zeta(x+v)$. Note that the statistical properties of the profile function, $\zeta(x)$, enter Eq. (9) only through $\Omega(v)$. For a Gaussian self-affine surface, one gets

$$\Omega(v)=\int_{-\infty}^{\infty}dz\,\exp\left\{-i\frac{\omega}{c}(\cos\theta+\cos\theta_0)z\right\}p(z;v) =\exp\left\{-\left(\frac{\omega}{c}\,\frac{\cos\theta+\cos\theta_0}{\sqrt{2}}l^{1-H}v^H\right)^2\right\}. \tag{11}$$

By a simple change of variable in Eq. (9) and letting the length of the profile extend to infinity, one finally obtains the following expression for the mean differential reflection coefficient:

$$\left\langle\frac{\partial R}{\partial\theta}\right\rangle=\frac{a^{-[(1/H)-1]}}{\sqrt{2}\cos\theta_0}\,\frac{\cos\frac{\theta+\theta_0}{2}}{\cos^3\frac{\theta-\theta_0}{2}}\mathcal{L}_{2H}\left(\frac{\sqrt{2}\tan\frac{\theta-\theta_0}{2}}{a^{(1/H-1)}}\right), \tag{12a}$$

where $a=\sqrt{2}(\omega/c)l[\cos(\theta+\theta_0)/2)][\cos(\theta-\theta_0)/2)]$, and $\mathcal{L}_\alpha(x)$ is the centered symmetrical Lévy stable distribution of exponent α defined as

$$\mathcal{L}_\alpha(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\quad e^{ikx}e^{-|k|^\alpha}. \tag{12b}$$

Note that in the above expressions, the wavelength λ only comes into play through the combination $(l/\lambda)^{1-(1/H)}$, which appears both in the prefactor and in the argument of the Lévy distribution. This quantity can be geometrically regarded as $s(\lambda)^{1/H}$, where $s(\lambda)=(l/\lambda)^{1-H}$ is the typical slope of the surface over a wavelength. The behavior of the scattered intensity is thus entirely determined by this typical slope $s(\lambda)$ and the roughness exponent H.

Taking advantage of the asymptotic expansion of the Lévy distribution around zero [23], we find that the amplitude of the specular peak scales as

$$\left\langle\frac{\partial R}{\partial\theta}\right\rangle\bigg|_{\theta=\theta_0}\simeq\frac{\Gamma\left(\frac{1}{2H}\right)}{2\sqrt{2}\pi H\left(2\sqrt{2}\pi\frac{l}{\lambda}\cos\theta_0\right)^{(1/H)-1}} \tag{13}$$

and the half-width scales [assuming $(\omega/c)l\cos\theta_0\ll 1$] as

$$w\simeq 2\frac{\sqrt{\Gamma\left(\frac{1}{2H}\right)}}{\sqrt{\Gamma\left(\frac{3}{2H}\right)}}\left(2\sqrt{2}\pi\frac{l}{\lambda}\cos\theta_0\right)^{(1/H)-1}. \tag{14}$$

It is interesting that these nontrivial scaling results can all be retrieved via simple dimensional arguments. Let us examine the intensity scattered in direction θ; in a naive Huyghens framework two different effects will compete to destroy the coherence of two source points on the surface: (i) the angular difference separating θ from the specular direction, (ii) the roughness. The two source points will interfere coherently if the distance between them is lower than the two length scales $d_{\mathrm{ang}}=\lambda/p$, where $p=\tan[(\theta-\theta_0)/2]$ is the slope of the isophase plane and d_{rough} is the typical length scale over which the height difference is of order λ. In case of a self-affine surface, $d_{\mathrm{rough}}=\lambda^{1/H}l^{1-1/H}$. Depending on the observation angle θ, the coherence will be controlled either by d_{rough} or by d_{ang}, which will correspond, respectively, to the peak or the tail of the intensity distribution. The width w of the peak can then be defined as the transition between these two regions and will be such that $d_{\mathrm{rough}}\simeq d_{\mathrm{ang}}$. This leads directly to the scaling of Eq. (14): $w\propto(l/\lambda)^{(1/H)-1}$.

Using the expansion of the Lévy distribution at infinity [24], the following expression for the diffuse intensity ($\theta\neq\theta_0$) may be obtained:

$$\left\langle\frac{\partial R}{\partial\theta}\right\rangle\simeq\frac{\Gamma(1+2H)\sin(\pi H)}{4\pi\cos\theta_0\left(\cos\frac{\theta+\theta_0}{2}\right)^{2H-3}}\frac{\left(4\pi\frac{l}{\lambda}\right)^{2-2H}}{\left|\sin\frac{\theta-\theta_0}{2}\right|^{1+2H}}. \tag{15}$$

Note that the mean DRC depends only on the roughness exponent and on the topothesy parameter l. We stress that in this single-scattering approach, the relevant parameter for the description of the roughness amplitude is the topothesy l and not the root mean square (rms) roughness. Two surfaces may thus shear the same roughness exponent and rms roughness value and differ by their topothesy (this may occur if the upper cutoff of the self-affine regime is different); such surfaces would present different scattering properties.

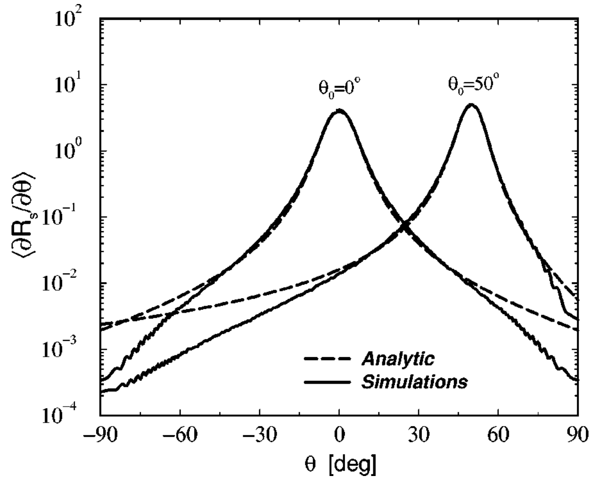

FIG. 2. The mean differential reflection coefficient for incidence angles $\theta_0=0°$ and $\theta_0=50°$ and a wavelength $\lambda=612.7$ nm as obtained from Eq. (12) (dashed lines) and by direct numerical simulations (solid lines). The surfaces each of length $L=100\lambda$ had topothesy $l=10^{-4}\lambda$ and a roughness exponent $H=0.7$. The height standard deviation as measured over the whole surface length was $\sigma(L)=1.6\,\lambda=1$ μm. The simulation results were averaged over $N_\zeta=1000$ surface realizations.

In order to test the analytic result (12), we have performed direct numerical simulations for the mean differential reflection coefficient for an s-polarized plane incident wave of wavelength $\lambda=2\pi c/\omega=612.7$ nm scattered from self-affine surfaces. Such simulations were carried out by the now quite standard extinction theorem technique [22], a numerical approach known to produce reliable results. In order to calculate the mean differential reflection coefficient, an en-

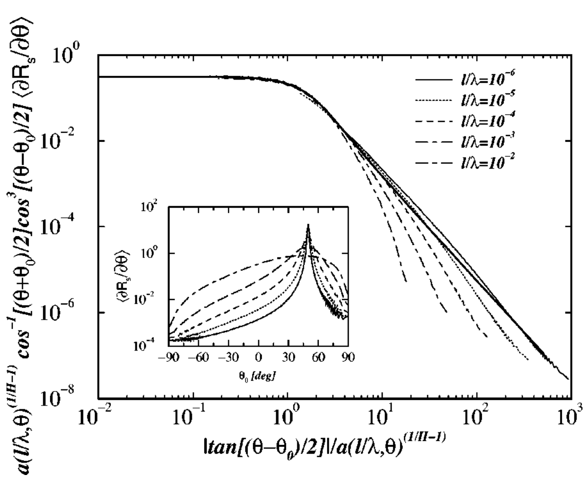

FIG. 3. Direct numerical simulation results for the mean DRC for surfaces of $l/\lambda=10^{-6},10^{-5},10^{-4},10^{-3}$, and 10^{-2} and an incident angle $\theta_0=50°$. The inset shows the bare data; in rescaled coordinates all data tend to collapse on a Lévy distribution of exponent $2H$ (bold line).

semble of Gaussian self-affine surfaces, all characterized by the same roughness exponent, H, and topothesy, l, was generated by the Fourier filtering technique [22], for which the results for $\langle \partial R/\partial \theta \rangle$ were averaged. The number of samples used in these averages was $N_\zeta = 1000$. For each surface the length was $L = 100\lambda$ and the spatial discretization length was $\Delta x \simeq \lambda/10$. For all numerical results to be presented, energy conservation was checked and found to be satisfied within at least 0.1% for all incident angles considered. Furthermore, the error due to the use of a finite numbers of samples was below 1%.

In Fig. 2 we present the numerical simulation results (solid lines) for the mean DRC for s-polarized light incident at the angles $\theta_0 = 0°$ and $\theta_0 = 50°$ and scattered from a perfectly conducting, Gaussian self-affine surface of Hurst exponent $H = 0.7$ and topothesy $10^{-4}\lambda$. These results are compared to the analytic predictions obtained from Eq. (12) (dashed lines). An excellent agreement between the analytical and numerical results is found for scattering angles of roughly 50°, or less, from the specular direction. For scattering angles outside this range, the analytical prediction is overestimated compared to the simulation results. This overestimation for grazing scattering angles is the expected signature of multiscattering and shadowing effects. Note that the surfaces considered above ($l = 10^{-4}\lambda$) are rather rough; calculated over the whole surface, the rms roughness amounts to $\sigma(L) = 1.45\lambda$.

In Fig. 3 we have gathered numerical results for the mean DRC obtained for an incidence angle $\theta_0 = 50°$ for surfaces of (topothesy)/(wavelength) $l/\lambda = 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}$, and 10^{-2}. The bare data are shown in the inset. After rescaling the coordinates, we observe that the data tend to collapse on a master curve which from Eq. (12) is a Lévy distribution for all values of l/λ. Deviations from the Lévy distribution occur for low incidence or scattering angles when multiple scattering and shadowing effects are expected to become important. They are beyond the scope of our theoretical analysis.

We considered the scattering of a wave by a perfectly conducting self-affine rough surface. Within the Kirchhoff approximation, we could compute analytically the angular distribution of the scattered intensity. We found an excellent agreement with direct numerical simulation obtained with rms roughness amplitudes of the order of several wavelengths.

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[23] At zero,

$$\mathcal{L}_\alpha(x) = \frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left[1 - \frac{\Gamma\left(\frac{3}{\alpha}\right)}{2\Gamma\left(\frac{1}{\alpha}\right)}x^2\right] + O(x^4).$$

[24] At infinity,

$$\mathcal{L}_\alpha(x) = \frac{\Gamma(1+\alpha)}{\pi|x|^{1+\alpha}}\sin\left(\frac{\alpha\pi}{2}\right) + O\left(\frac{1}{|x|^{1+2\alpha}}\right).$$